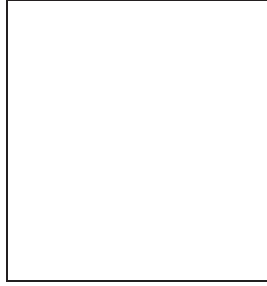


Non-relativistic Quantum Mechanics versus Quantum Field Theories

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We briefly review the derivation of a non-relativistic quantum mechanics description of a weakly bound non-relativistic system from the underlying quantum field theory. We highlight the main techniques used.

In a first approximation, the dynamics of the Hydrogen atom can be described by the solution of the Schrödinger equation with a Coulomb potential. However, it is not always clear how to derive this equation from the more fundamental quantum field theory, QED, much less how to get corrections in a systematic way. A similar problem is faced in heavy quarkonium systems with very large heavy quark masses. In this situation the dynamics is mainly perturbative and one efficient solution to this problem comes from the use of effective field theories (EFTs) and in particular of pNRQCD^{1a}. This EFT takes full advantage of the hierarchy of scales that appear in the system (v is the velocity of the heavy quark in the center of mass frame and m is the heavy quark mass):

$$m \gg mv \gg mv^2 \dots \quad (1)$$

and makes systematic and natural the connection of the Quantum Field Theory with the Schrödinger equation. Roughly speaking the EFT turns out to be something like:

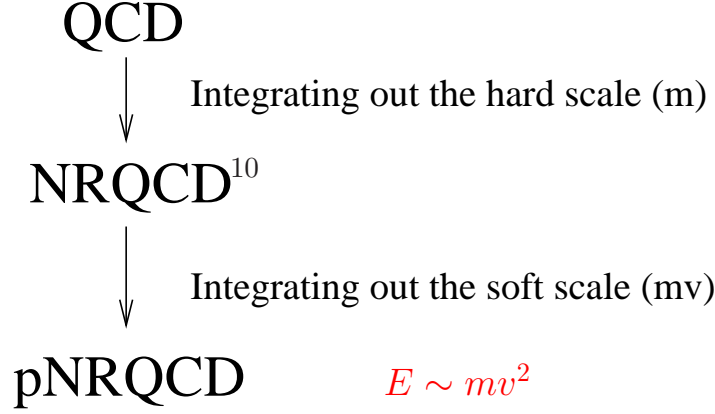
$$\left(i\partial_0 - \frac{\mathbf{p}^2}{2m} - V_s^{(0)}(r) \right) \Phi(\mathbf{r}) = 0 + \text{corrections to the potential} \Bigg\} \text{pNRQCD} \\ + \text{interaction with other low - energy degrees of freedom}$$

where $V_s^{(0)}(r) = -C_f \alpha_s / r$ in the perturbative case and $\Phi(\mathbf{r})$ is the \bar{Q} - Q wave-function. This

^aFor a comprehensive review of pNRQCD see².

EFT^b is relevant, at least, for the study of the ground state properties of the bottomonium system, non-relativistic sum rules and the production of $t\bar{t}$ near threshold (for some recent applications see^{5,6,7,8,9}).

The key point in the construction of the EFT is to determine the kinematic situation we want to describe. This fixes the (energy of the) degrees of freedom that appear as physical states (and not only as loop fluctuations). In our case the degrees of freedom in pNRQCD are kept to have $E \sim mv^2$. In order to derive pNRQCD we sequentially integrate out the larger scales.



In this paper, we would like to highlight the main techniques needed in order to perform efficiently high-precision perturbative computations in non-relativistic bound state systems. They can be summarized in four points:

1. Matching QCD to NRQCD: Relativistic Feynman diagrams
2. Matching NRQCD to pNRQCD (getting the potential): Non-Relativistic (HQET-like) Feynman diagrams
3. Observable: Quantum mechanics perturbation theory
4. Observable: Ultrasoft loops

The first two points explain the techniques needed to obtain pNRQCD from QCD, whereas the last two explain the kind of computations faced in the EFT when computing observables. All the computations can be performed in dimensional regularization and only one scale appears in each type of integral, which becomes homogeneous. This is a very strong simplification of the problem. In practice this is implemented in the following way:

Point 1). One analytically expands over the three-momentum and residual energy in the integrand before the integration is made in both the full and the effective theory^{11,12}.

$$\begin{aligned}
 QCD \quad \int d^4q f(q, m, |\mathbf{p}|, E) &= \int d^4q f(q, m, 0, 0) + \mathcal{O}\left(\frac{E}{m}, \frac{|\mathbf{p}|}{m}\right) \sim C\left(\frac{\mu}{m}\right)(\text{tree level})|_{NRQCD} \\
 NRQCD \quad \int d^4q f(q, |\mathbf{p}|, E) &= \int d^4q f(q, 0, 0) = \mathbf{0}!!
 \end{aligned} \tag{2}$$

Therefore, the computation of loops in the effective theory just gives **zero** and **one matches loops in QCD with only one scale (the mass) to tree level diagrams in NRQCD**, which we schematically draw in the following figure:

^bIt is also possible to study heavy quarkonium systems in the non-perturbative regime with pNRQCD profiting from the hierarchy of scales of Eq. (1), see^{3,4}.

$$\begin{aligned}
& \text{QCD Box Diagram} = \frac{C(m/\mu)}{m^2} \text{NRQCD Box Diagram} + \mathcal{O}(1/m^2) \\
& \text{QCD Triangle Diagram} = \frac{C(m/\mu)}{m} \text{NRQCD Triangle Diagram} + \dots
\end{aligned}$$

OCD
NROCD

Point 2) works analogously¹³. One expands in the scales that are left in the effective theory. We integrate out the scale k (transfer momentum between the quark and antiquark). Again loops in the EFT are zero and only tree-level diagrams have to be computed in the EFT:

$$\text{NRQCD} \quad \int d^4 q f(q, k, |\mathbf{p}|, E) = \int d^4 q f(q, k, 0, 0) + \mathcal{O}\left(\frac{E}{k}, \frac{|\mathbf{p}|}{k}\right) \sim \delta h_s(\text{potential}) \quad (3)$$

$$\text{pNRQCD} \quad \int d^4 q f(q, |\mathbf{p}|, E) = \int d^4 q f(q, 0, 0) = 0!! \quad (4)$$

We illustrate the matching in the figure below. Formally the one-loop diagram is equal to the QCD diagram shown above. The difference is that it has to be computed with the HQET quark propagator ($1/(q^0 + i\epsilon)$) and the vertices are also different.

$$\begin{aligned}
& \text{NRQCD Box Diagram} \sim \frac{\alpha}{k^2} = \text{pNRQCD Box Diagram} \\
& \text{NRQCD Triangle Diagram} \sim \frac{\alpha^2}{m^2} (\ln k + c) = \text{pNRQCD Triangle Diagram}
\end{aligned}$$

NRQCD
pNRQCD

Once the Lagrangian of pNRQCD has been obtained one can compute observables. A key quantity in this respect is the Green function. In order to go beyond the leading order description of the bound state one has to compute corrections to the Green Function (δh_s schematically represents the corrections to the potential and H_I the interaction with ultrasoft gluons):

$$G_s(E) = \frac{1}{h_s^{(0)} + \delta h_s - H_I - E} = G_s^{(0)} + \delta G_s \quad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E} .$$

These corrections can be organized as an expansion in $1/m$, α_s and the **multipole expansion**. Two type of integrals appear then, which correspond to points 3) and 4) above.

Point 3). For example, if we were interested in computing the spectrum at $O(m\alpha_s^6)$ (for QED see¹⁴), one should consider the iteration of subleading potentials (δh_s) in the propagator:

$$\delta G_s^{pot.} = \text{---} \overline{\square} \text{---} + \text{---} \overline{\square} \overline{\square} \text{---} + \dots$$

$$\sim \frac{1}{h_s^{(0)} - E} \delta h_s \frac{1}{h_s^{(0)} - E} + \frac{1}{h_s^{(0)} - E} \delta h_s \frac{1}{h_s^{(0)} - E} \delta h_s \frac{1}{h_s^{(0)} - E} + \dots$$

At some point, these corrections produce divergences. For example, a correction of the type: $\delta(r) G_s^{(0)}(C_f \alpha_s / r) G_s^{(0)} \delta(r)$, would produce the following divergence

$$\langle \mathbf{r} = 0 | \frac{1}{E - \mathbf{p}^2/m} C_f \frac{\alpha_s}{r} \frac{1}{E - \mathbf{p}^2/m} | \mathbf{r} = 0 \rangle$$

$$\sim \int \frac{d^d p'}{(2\pi)^d} \int \frac{d^d p}{(2\pi)^d} \frac{m}{\mathbf{p}^2 - mE} C_f \frac{4\pi\alpha_s}{(\mathbf{p} - \mathbf{p}')^2} \frac{m}{\mathbf{p}^2 - mE} \sim -C_f \frac{m^2\alpha_s}{16\pi} \left(\frac{1}{\epsilon} + 2\ln\left(\frac{mE}{\mu_p}\right) + \dots \right).$$

Nevertheless, the existence of divergences in the effective theory is not a problem since they get absorbed in the potentials (δh_s). The same happens with ultrasoft gluons, **point 4**)^{15,16}:

$$\delta G_s^{us} = \text{---} \overbrace{\text{---} \otimes \text{---}}^{1/(E - V_o^{(0)} - \mathbf{p}^2/m)} \text{---} \sim G_c(E) \int \frac{d^d \mathbf{k}}{(2\pi)^d} \mathbf{r} \frac{k}{k + \mathbf{p}^2/m + V_o^{(0)} - E} \mathbf{r} G_c(E)$$

$$\sim G_c(E) \mathbf{r} (\mathbf{p}^2/m + V_o^{(0)} - E)^3 \left\{ \frac{1}{\epsilon} + \gamma + \ln \frac{(\mathbf{p}^2/m + V_o^{(0)} - E)^2}{\nu_{us}^2} + C \right\} \mathbf{r} G_c(E),$$

which also produces divergences that get absorbed in δh_s . Overall, we get a consistent EFT.

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